Barrier in Pursuit-Evasion Problems between Two Low-Thrust Orbital Spacecraft

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In a zero-sum free-time differential game, the barrier is an $n-1$ dimensional surface in n-dimensional state space. On one side of the barrier, the minimizing player can force termination of the game to an algebraic terminal surface by playing a suitable optimal strategy, whereas on the other side the maximizing player, by playing an optimal strategy, can avoid termination at least in the short term. This paper presents a closed-form solution for the barrier in a planar pursuit-evasion problem between two low-thrust orbital spacecraft. This solution is obtained by linearizing the problem about a reference circular orbit and approximating the thrust angle control by a polynomial in time-to-go. The region of validity of this approximate solution is checked by comparing it with numerically generated solutions for points on the barrier for the complete nonlinear problem. A method is presented for determining on which side of the barrier a given state is located. This closed-form solution then is used as the basis for a proposed feedback guidance law for orbital interception or evasion problems between low-thrust spacecraft.

I. Introduction

ALTHOUGH optimal rocket trajectories for orbital transfer and rendezvous have been studied extensively by many authors, the problem of optimal interception of one maneuvering space vehicle by another has received relatively little attention. The only work known to the authors treating this subject are the papers by Wong,[1] Billik,[2] and Fitzgerald,[3] all of whom analyze the subject using zero-sum differential game theory.[4] The interception problems treated in these papers are all formulated as "games of degree"; that is, one player attempts to minimize some specified payoff J while the other player attempts to maximize it, subject to some terminal condition being satisfied. The problems considered by Wong are minimax range with final time fixed and minimax time to zero final range (i.e., interception). The only interception-type problem considered by Billik is a fixed-time, fixed-final-range problem with a passive evader. Fitzgerald examines the problem of planar intercept of a deorbiting space vehicle with interception ΔV as the payoff. All of these papers treat highly simplified problems assuming linear dynamics, with no attempt being made to analyze the extent of the validity of the simplifying assumptions.

This paper treats the problem of the "game of kind" where one orbital spacecraft, with low constant thrusting acceleration, attempts to intercept another spacecraft of similar type, where interception is defined to occur when the pursuing vehicle gets within a given distance of the evading vehicle. There is a surface called the barrier in the position-velocity state space of the game which divides the space into two regions: 1) a region from which the pursuer is guaranteed that, by playing a suitable optimal strategy, he can attain interception of the evader, and 2) a region from which the evader is guaranteed that, by playing a suitable optimal strategy, he can avoid interception, at least in the short term. In general, this barrier is not a closed surface, so that, given a sufficiently long time period and sufficient fuel, the pursuer eventually could intercept the evader. However, here we shall be concerned only with the short-term problem of interception or escape on the "first pass."

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The main result in this paper is a near-optimal closed-form solution for the barrier which is obtained using linearized dynamics and a polynomial approximation in terms of time-to-go for the optimal thrusting angles. The region of validity of this solution is checked by comparing its results with numerically generated points on the barrier using the complete nonlinear orbital dynamics. An algebraic search scheme also is presented which determines whether a given state is on one side of the barrier or the other, and the possible application of these results to an interception guidance law is discussed.

In Sec. II, the mathemetical conditions for finding barriers are summarized. The general nonlinear problem is treated first in Sec. III. The problem then is linearized and solved to obtain the near-optimal closed-form barrier solution in Sec. IV. The results from the nonlinear and linear barrier analyses also are compared here to check the region of validity of the near-optimal closed-form solutions. In Sec. V, an algebraic iteration scheme is presented to determine whether a given state is on one side of the barrier or the other. In Sec. VI, these results are used to establish a feedback control law, which can be used by either player to take advantage of any nonoptimal play of the other. Conclusions are presented in Sec. VII followed by an Appendix, which contains the details of the near-optimal closed-form solution for the barrier.

II. Barrier in Differential Games

To introduce the concept of the barrier in differential games and methods for finding this barrier, consider the following general free-time, perfect information, zero-sum differential game. The autonomous vector state equation is

$$\dot{x}=f(x,u,v) \tag{1}$$

where x is an n vector, u is the vector control for the minimizing player (the pursuer p), and v is the vector control for the maximizing player (the evader e). Both u and v may be subject to inequality constraints, which, for simplicity, are assumed to be independent of x. The game starts from some initial state

$$x(t_0)=x_0 \tag{2}$$

and the game terminates when the following algebraic terminal condition is satisfied:

$$\psi[x(t_f)]-K=0 \tag{3}$$

Such differential games have payoffs of the general form

$$J=\phi[(t_f)]+\int_{t_0}^{t_f} L(x,u,v)\,dt \tag{4}$$

The pursuer p desires to choose his control vector u so as to minimize J with all of the previous conditions being satisfied. Conversely, the evader e desires to choose his control v so as to maximize J. If u^* and v^* denote the optimal controls for the two players, the differential game saddle point condition can be expressed by the inequality

$$J(u^*,v) \leq j(u^*,v^*) \leq J(u,v^*) \tag{5}$$

where u and v are any admissible controls for the respective players. This, of course, assumes that a saddle point for the game exists.

Let us assume that the game is structured so that it is to p's advantage to force the game to termination and to e's advantage to avoid termination. This is a common situation in pursuit-evasion combat games in which the terminal condition given by Eq. (3) represents a "kill" distance for p's weapon system. Regardless of the form of J, if p can force the game to termination, he wins. This situation can be interpreted as a "Game of kind," which is not concerned with a payoff J but only whether or not p can force the state of the game to satisfy Eq. (3) at some unspecified t_f.

Following Isaacs,[4] the terminal surface defined by Eq. (3) can, in general, be divided into three parts. First define $\lambda(t_f)$ to be an n vector normal to the surface $\psi - K = 0$ pointing outward toward the region in which the game is played. If the following condition holds just off the terminal surface, p can force immediate termination of the game

$$\min_u \max_v \lambda^T(t_f) f[x(t_f),u,v] < 0 \tag{6}$$

Conversely, e can avoid immediate termination if

$$\min_u \max_v \lambda^T(t_f) f[x(t_f),u,v] > 0 \tag{7}$$

The borderline condition is

$$\min_u \max_v \lambda^T(t_f) f[x(t_f),u,v] = 0 \tag{8}$$

This equation defines the "boundary of the usable part" (BUP) of the terminal surface and represents the locus of points on the terminal surface on which the barrier terminates.

To generate the barrier itself, define the "barrier Hamiltonian" by

$$H_b = \lambda^T f(x,u,v) \tag{9}$$

The differential equation for the barrier costate vector $\lambda(t)$ is

$$\dot{\lambda}^T = H_{b_x} \tag{10}$$

with condition at t_f that $\lambda(t_f)$ be normal to the terminal surface on the BUP. One convenient expression for $\lambda(t_f)$ is

$$\lambda^T(t_f) = \psi_x[x(t_f)] \tag{11}$$

where $\psi_x[x(t_f)]$ is evaluated at points on the BUP. Along the barrier, the optimal u (denoted by u^*) is obtained by minimizing H_b with respect to u, and the optimal v (denoted by v^*) is found by maximizing H_b with respect to v. Because of the assumption that the system is autonomous, the following relationship must hold along the barrier:

$$H_b = \lambda^T f(x,u^*,v^*) = 0 \tag{12}$$

This simply states that $\lambda(t)$ always is normal to the state velocity vector along a barrier trajectory.

Backward integration of Eqs. (1) and (10) from points on the BUP will generate barrier trajectories. The union of all of the barrier trajectories from the BUP comprises the barrier itself. Note that, if x has dimension n, then the barrier has dimension $n-1$, and the BUP has dimension $n-2$. For $n>3$, the barrier is extremely difficult to depict graphically or analytically.

There is another interpretation of the barrier which will prove useful later. Consider a differential game with $x(t_f)$ and t_f free, in which the goal of the pursuer to minimize $\psi[x(t_f)]$ {this is the same $\psi[x(t_f)]$ that appears in Eq. (3)} and e strives to maximize this quantity. We further assume that $\psi[x(t_f)]$ has a well-defined minimum, say $\psi[x(t_f)]=0$, which is the optimum terminal position for p. Application of the necessary saddle point conditions for this problem again gives Eqs. (9-12). If we consider all terminal states such that $\psi[x(t_f)]=K$, backward integration of the differential equations again will produce the barrier trajectories. Thus the barrier can be considered to consist of those initial conditions such that play of a minimax ψ strategies results in a value of $\psi x(t_f)]=K$. In this context, we can consider $\psi[x(t_f)]$ to represent a generalized range function.

III. Nonlinear Orbital Barrier Problem

Consider a planar pursuit-evasion differential game between two low-thrust space vehicles whose positions and velocities are close to those of a reference circular orbit. In terms of position, velocity, and time variables normalized with respect to this reference circular orbit, the equations of motion for each vehicle are

$$\dot{r} = V_r \tag{13a}$$

$$\dot{V}_r = V_\theta^2/r - 1/r^2 + T\sin\alpha \tag{13b}$$

$$\dot{V}_\theta = -V_r V_\theta / r + T\cos\alpha \tag{13c}$$

$$\dot{\theta} = V_\theta / r \tag{13d}$$

where r is the distance from the center of an inverse square gravitational field, θ is an angular position, V_r and V_θ are velocity components, T is a small constant thrust magnitude per unit mass, and α is the thrust direction relative to the local horizontal direction. The thrust magnitude T is assumed to be sufficiently small so that the rocket mass can be considered constant during the game. The game terminates when the pursuer approaches to within a specified distance R_0 from the evader, or

$$\psi = [(r_e - r_p)^2 + (r_e\theta_e - r_p\theta_p)^2 - R_0^2]/2 = 0 \tag{14}$$

where the subscript e denotes the evader and p the pursuer. This expression can be simplified considerably if r_e and r_p both are assumed to be very close to the reference radius of $r_0 = 1$. Then Eq. (14) reduces to

$$\psi = [(r_e - r_p)^2 + (\theta_e - \theta_p)^2 - R_0^2]/2 = 0 \tag{15}$$

At the BUP, the λ vector is normal to the terminal surface. Defining $\lambda(t_f)$ to be a unit vector, we have, using Eq. (11) and normalizing the result,

$$\lambda_{re}(t_f) = -\lambda_{rp}(t_f) = \sin\phi \tag{16a}$$

$$\lambda_{\theta e}(t_f) = -\lambda_{\theta p}(t_f) = \cos\phi \tag{16b}$$

$$\lambda_{Vre}(t_f) = \lambda_{Vrp}(t_f) = 0 \tag{16c}$$

$$\lambda_{V\theta e}(t_f) = \lambda_{V\theta p}(t_f) = 0 \tag{16d}$$

where ϕ is defined by

$$\sin\phi = (r_e - r_p)/R_0, \quad \cos\phi = (\theta_e - \theta_p)/R_0 \tag{17}$$

Using Eqs. (8, 13, and 16), we obtain the following expression for the BUP

$$(V_{re} - V_{rp})\sin\phi + (V_{\theta e} - V_{\theta p})\cos\phi = 0 \tag{18}$$

where, once again, we have assumed that r_e and r_p are very close to the reference $r_0 = 1$.

In addition to the eight state differential equations, there are eight costate differential equations for the λ vector components. The equations for the components associated with the pursuer's state variables have exactly the same form as those for the evader. For each player, these equations are

$$\dot{\lambda}_r = (\lambda_{Vr} V_\theta^2 - 2\lambda_{Vr}/r - \lambda_{V\theta} V_r V_\theta + \lambda_\theta V_\theta)/r^2 \qquad (19a)$$

$$\dot{\lambda}_{Vr} = -\lambda_r + \lambda_{V\theta} V_\theta / r \qquad (19b)$$

$$\dot{\lambda}_{V\theta} = (-2\lambda_{Vr} V_\theta + \lambda_{V\theta} V_r - \lambda_\theta)/r \qquad (19c)$$

$$\dot{\lambda}_\theta = 0 \qquad (19d)$$

The expressions for the thrust direction for each player along the barrier are given by

$$\sin\alpha = \lambda_{Vre} / (\lambda_{Vre}^2 + \lambda_{V\theta e}^2)^{1/2} \qquad (20a)$$

$$\cos\alpha = \lambda_{V\theta e} / (\lambda_{Vre}^2 + \lambda_{V\theta e}^2)^{1/2} \qquad (20b)$$

These expressions are undefined on the BUP, since, from Eqs. (16), the velocity costates are all zero at this point. However, application of l'Hospital's rule along with Eqs. (16) gives, at the terminal surface,

$$\sin\alpha_e = \sin\alpha_p = -\sin\phi \qquad (21a)$$

$$\cos\alpha_e = \cos\alpha_p = -\cos\phi \qquad (21b)$$

where, in applying l'Hospital's rule, we again have assumed that $r_p(t_f)$ and $r_e(t_f)$ are very close to the reference radius of $r_0 = 1$.

Trajectories on the barrier now can be generated by numerically integrating the 16 differential equations given by Eqs. (13) and (19) (for both players) from points on the BUP with Eqs. (16) satisfied. The union of all barrier trajectories gives the barrier itself. Since the barrier is a seven-dimensional surface in the eight-dimensional state space, it is almost impossible to analyze the barrier using this general nonlinear approach.

IV. Linearized Orbital Barrier Problem

Because of the difficulty of obtaining any useful information about the barrier from the nonlinear analysis of Sec. III, an approximate closed-form solution for this barrier is derived in this section. First the equations of motion are linearized in state about the reference circular orbit, and the state space is reduced to dimension four by writing the linearized motion of the evader relative to the pursuer. Following the application of the barrier necessary conditions, the trigonometric functions of the thrust angle control are expanded in a polynomial series in time-to-go in order to get the state equations into a form that can be integrated in closed form. Two tests to check the validity of the resulting algebraic barrier solutions then are presented, followed by a comparison of some algebraic solutions with the corresponding numerically generated trajectories for the complete nonlinear problem.

Linearization of State Equations

The linearized equations are to be referenced to a circular orbit with radius $r_0 = 1$ and circular velocity $V_0 = 1$. Let us define the following variables

$$\Delta r = r - r_0, \quad \Delta V_r = V_r \qquad (22a)$$

$$\Delta V_\theta = V_\theta - V_0, \quad \Delta\theta = \theta - \theta_0 \qquad (22b)$$

In terms of these variables, linearization of Eq. (13) results in

$$\Delta\dot{r} = V_r \qquad (23a)$$

$$\Delta\dot{V}_r = 2\Delta V_\theta + \Delta r + T\sin\alpha \qquad (23b)$$

$$\Delta\dot{V}_\theta = -\Delta V_r + T\cos\alpha \qquad (23c)$$

$$\Delta\dot{\theta} = \Delta V_\theta - \Delta r \qquad (23d)$$

Note that these equations are not linear in the control α.

Reduction of Order of Problem

Recall that the nonlinear development of Sec. III required eight state equations. For the linearized problem, the state dimension can be reduced to four by defining the following variables, which describe the motion of the evader relative to the pursuer

$$Y_1 = \Delta r_e - \Delta r_p, \quad Y_2 = \Delta V_{re} - \Delta V_{rp} \qquad (24a)$$

$$Y_3 = \Delta V_{\theta e} - \Delta V_{\theta p}, \quad Y_4 = \Delta\theta_e - \Delta\theta_p \qquad (24b)$$

The resulting differential equations are

$$\dot{Y}_1 = Y_2 \qquad (25a)$$

$$\dot{Y}_2 = 2Y_3 + Y_1 + T_e \sin\alpha_e - T_p \sin\alpha_p \qquad (25b)$$

$$\dot{Y}_3 = -Y_2 + T_e \cos\alpha_e - T_p \cos\alpha_p \qquad (25c)$$

$$\dot{Y}_4 = Y_3 - Y_1 \qquad (25d)$$

In terms of these new variables, the terminal condition can be expressed as

$$\psi = (Y_1^2 + Y_4^2 - R_0^2)/2 = 0 \qquad (26)$$

Linearized Barrier Necessary Conditions

A unit vector normal to the terminal surface is defined by

$$p_1 = Y_1 / R_0 = \sin\phi \qquad (27a)$$

$$p_2 = p_3 = 0 \qquad (27b)$$

$$p_4 = Y_4 / R_0 = \cos\phi \qquad (27c)$$

where the p_i are the barrier costates associated with this linearized problem.

The barrier costate differential equations are

$$\dot{p}_1 = -p_2 + p_4 \qquad (28a)$$

$$\dot{p}_2 = -p_1 + p_3 \qquad (28b)$$

$$\dot{p}_3 = -2p_2 - p_4 \qquad (28c)$$

$$\dot{p}_4 = 0 \qquad (28d)$$

The controls for the two players are

$$\sin\alpha_e = \sin\alpha_p = p_2 / (p_2^2 + p_3^2)^{1/2} \qquad (29a)$$

$$\cos\alpha_e = \cos\alpha_p = p_3 / (p_2^2 + p_3^2)^{1/2} \qquad (29b)$$

The BUP for the linearized problem is described by

$$[Y_1 Y_2 + Y_4 (Y_3 - Y_1)]_{t_f} / R_0 = 0 \qquad (30)$$

or

$$[Y_2 \sin\phi + Y_3 \cos\phi - R_0 \sin\phi \cos\phi]_{t_f} = 0 \qquad (31)$$

Integration of Eqs. (25) and (28) backwards from the BUP, with Eqs. (26) and (27) satisfied and the controls α_e and α_p given by Eq. (29), will generate barrier trajectories for the linearized problem. Even though the costate equations can be integrated in closed form, no closed-form solution for the states could be found because of the nonlinearity of the control terms.

Control Approximation and Barrier Solution

In terms of a time-to-go variable τ measured backwards from the BUP, the solution for the costates is

$$p_1 = -\sin\phi \cos\tau + 2\cos\phi \sin\tau - 3\tau\cos\phi + 2\sin\phi \quad (32a)$$

$$p_2 = \sin\phi \sin\tau + 2\cos\phi\cos\tau - 2\cos\phi \quad (32b)$$

$$p_3 = -2\sin\phi\cos\tau + 4\cos\phi\sin\tau - 3\tau\cos\phi + 2\sin\phi \quad (32c)$$

$$p_4 = \cos\phi \quad (32d)$$

To get the state equations into a form that can be integrated in closed form, the preceding expressions for p_2 and p_3 were substituted into Eqs. (29), and the results were expanded in a Taylor series about $\tau = 0$ (the conditions at the BUP). Neglecting terms of order τ^4, the approximate expressions for the controls are

$$\sin\alpha_e = \sin\alpha_p = \sin\phi - \tau\cos\phi - \tau^2 \sin^3\phi/2 - \tau^3\cos\phi/12 \quad (33)$$

$$\cos\alpha_e = \cos\alpha_p = \cos\phi + \tau\sin\phi - \tau^2\cos\phi(1+\sin^2\phi)/2 - \tau^3\sin\phi(5/6 - \cos^2\phi)/2 \quad (34)$$

Substitution of these control approximations into the state equations, Eqs. (25), results in a set of linear constant coefficient nonhomogeneous first-order differential equations. The backward solution to these equations is straightforward, although tedious, and the results are lengthy. For this reason, the complete closed-form solution is presented in the Appendix. These solutions have the following general form.

$$Y_1 = Y_2(0) f_{1Y}(\tau,\phi) + R_0 f_{1R}(\tau,\phi) + f_1(\tau,\phi) \quad (35a)$$

$$Y_2 = Y_2(0) f_{2Y}(\tau,\phi) + R_0 f_{2R}(\tau,\phi) + f_2(\tau,\phi) \quad (35b)$$

$$Y_3 = Y_2(0) f_{3Y}(\tau,\phi) + R_0 f_{3R}(\tau,\phi) + f_3(\tau,\phi) \quad (35c)$$

$$Y_4 = Y_2(0) f_{4Y}(\tau,\phi) + R_0 f_{4R}(\tau,\phi) + f_4(\tau,\phi) \quad (35d)$$

where the $f_i(\tau,\phi)$ are nonlinear functions of τ and ϕ. Note that these solutions are linear in the radial velocity $Y_2(0)$ at the BUP and the radius R_0 of the terminal surface. Unfortunately, they are highly nonlinear in τ and ϕ.

Validity Tests for Barrier Solution

Care must be taken in using Eqs. (35) to generate barrier trajectories backwards from the BUP because of spurious solutions that may arise in two ways. First, for the barrier trajectory to be valid, the two parameters at the terminal surface ϕ and $Y_2(0)$ must be such that the barrier does not pass inside the terminal surface for small values of τ. To derive a test to check for this situation, define the range r by

$$r^2 = Y_1^2 + Y_4^2 \quad (36)$$

By using Eqs. (35) to substitute for Y_1 and Y_2, an expansion of Eq. (36) in a Taylor series to order τ^2 yields

$$r^2 = R_0^2 + \tau^2[V_0^2 - 2R_0V_0 - (T_p - T_e)R_0 + 3R_0^2\sin^2\phi] \quad (37)$$

where the velocity V_0 at the terminal surface is defined by

$$V_0 = Y_2(0)/\cos\phi \quad (38)$$

The trajectory will remain outside the terminal surface for small τ if $r^2 > R_0^2$ or

$$V_0^2 - 2R_0V_0 - (T_p - T_e)R_0 + 3R_0^2\sin^2\phi > 0 \quad (39)$$

This inequality always must be satisfied at the terminal surface.

A second possible source of spurious solutions from Eqs. (35) is that each barrier trajectory terminates at some value of τ, but Eqs. (35) continue to generate solutions for larger values of τ even though these results are meaningless. Physically, the termination of the barrier means that, if the pursuer has a greater thrusting capability than the evader, $T_p > T_e$, and the initial relative distance between the vehicles is sufficiently large, the pursuer always can capture the evader. To find an approximate expression for the value of τ at which the barrier ends, note the similarity between the linearized game and Isaacs' isotropic rocket problem.[4] For very small values of τ, when gravity has had little effect on the trajectories, the two problems are essentially the same if we assume no acceleration capability for the evader in the isotropic rocket problem. On page 249 of Ref. 4, Isaacs derives an expression for the value of τ when the barrier trajectory ends. In terms of the parameters of this problem, this result is

$$\tau_c = \{2[V_0^2 - R_0(T_p - T_e)]\}^{1/2}/(T_p - T_e) \quad (40)$$

Attempts to derive an exact expression for τ_c using Eqs. (35) were unsuccessful, probably because of the use of the polynomial approximation for the saddle point controls.

To illustrate the physical meaning of results with $\tau > \tau_c$, Eqs. (35) were solved for the values of τ, ϕ, $Y_2(0)$, and R_0 which are associated with a barrier trajectory passing through the state

$$Y_1 = -0.00275, \quad Y_2 = 0.01198, \quad Y_3 = 0.00734, \quad Y_4 = 0.00580 \quad (41)$$

Two solutions of Eqs. (35) resulted for $T_p - T_e = 0.04$. One gave $\tau = 0.3$, $\phi = 45°$, $Y_2(0) = 0.005$, and $R_0 = 4.78 \times 10^{-5}$. For these parameters, Eq. (40) gives $\tau_c = 0.2452 < \tau$, which indicates that this solution probably is not valid. However, the other solution for this state is $\tau = 0.261$, $\phi = 31.78°$, $Y_{20} = 0.0079$, $R_0 = 3.80 \times 10^{-5}$, with a τ_c from Eq. (40) of 0.325 $> \tau$. Note that this second solution has a smaller value of R_0 than the first. This indicates that, if $R_0 = 4.78 \times 10^{-5}$, capture occurs with a resulting minimax range of 3.80×10^{-5}, and the state given by Eqs. (41) is inside the barrier associated with the terminal surface $R_0 = 4.78 \times 10^{-5}$. These results show that Eq. (40) gives a good measure of the value of τ beyond which Eqs. (35) give spurious solutions.

Accuracy of Closed-Form Barrier Solution

In order to check the accuracy of Eqs. (35), solutions as functions of τ were compared to numerically generated barrier trajectories using the nonlinear barrier necessary conditions of Sec. 3 from a few points on the BUP. Table 1 shows the results of this comparison for $\phi = 45°$, $R_0 = 4.78 \times 10^{-5}$ (which is about 1000 ft for a low Earth orbit), $Y_2(0) = 5 \times 10^{-3}$, and $T_p - T_e = 0.04$. Part A of Table 1 gives the results using the linearized analysis, part B gives the nonlinear results for $T_p = 0.04$ and $T_e = 0$ (i.e., a nonmaneuvering evader), and part C shows the effects of the larger specific thrusts of $T_p = 0.15$ and $T_2 = 0.11$. For the nonlinear results, the Y_i are defined by

$$Y_1 = r_e - r_p, \quad Y_2 = V_{re} - V_{rp} \quad (42a)$$

$$Y_3 = V_{\theta e} - V_{\theta p}, \quad Y_4 = \theta_e - \theta_p \quad (42b)$$

Applying Eq. (40) to the parameters at the terminal surface used in generating the data, the barrier trajectory terminates at about $\tau_c = 0.2452$, so that the last two entries in the tables

Table 1 Check of accuracy of closed-form solution for $\phi = 45°$, $R_0 = 4.78 \times 10^{-5}$, $Y_2(0) = 5 \times 10^{-3}$, and $T_p - T_e = 0.04$

τ	Y_1	Y	Y_3	Y_4
		A) Results from closed-form solution		
0	3.380×10^{-5}	5.000×10^{-3}	-4.966×10^{-3}	3.380×10^{-5}
0.05	-2.620×10^{-4}	6.795×10^{-3}	-3.222×10^{-3}	2.338×10^{-4}
0.10	-6.417×10^{-4}	8.352×10^{-3}	-1.328×10^{-3}	3.259×10^{-4}
0.15	-1.093×10^{-3}	9.659×10^{-3}	6.952×10^{-4}	2.991×10^{-4}
0.20	-1.603×10^{-3}	1.071×10^{-2}	2.829×10^{-3}	1.442×10^{-4}
0.25	-2.159×10^{-3}	1.148×10^{-2}	5.053×10^{-3}	-1.464×10^{-4}
0.3	-2.747×10^{-3}	1.198×10^{-2}	7.344×10^{-3}	-5.786×10^{-4}
		B) Nonlinear Results ($T_p = 0.04$, $T_e = 0$)		
0	3.380×10^{-5}	5.000×10^{-3}	-4.966×10^{-3}	3.380×10^{-5}
0.05	-2.620×10^{-4}	6.793×10^{-3}	-3.224×10^{-3}	2.339×10^{-4}
0.10	-6.416×10^{-4}	8.350×10^{-3}	-1.331×10^{-3}	3.262×10^{-4}
0.15	-1.093×10^{-3}	9.660×10^{-3}	6.924×10^{-4}	2.997×10^{-4}
0.20	-1.603×10^{-3}	1.071×10^{-2}	2.830×10^{-3}	1.452×10^{-4}
0.25	-2.160×10^{-3}	1.150×10^{-2}	5.064×10^{-3}	-1.453×10^{-4}
0.3	-2.749×10^{-3}	1.202×10^{-2}	7.376×10^{-3}	-5.781×10^{-4}
		C)Nonlinear results ($T_p = 0.15$, $T_e = 0.11$)		
0	3.380×10^{-5}	5.000×10^{-3}	-4.966×10^{-3}	3.380×10^{-5}
0.05	-2.620×10^{-4}	6.792×10^{-3}	-3.223×10^{-3}	2.339×10^{-4}
0.10	-6.414×10^{-4}	8.347×10^{-3}	-1.333×10^{-3}	3.264×10^{-4}
0.15	-1.093×10^{-3}	9.657×10^{-3}	6.858×10^{-4}	3.006×10^{-4}
0.20	-1.603×10^{-3}	1.072×10^{-2}	2.813×10^{-3}	1.479×10^{-4}
0.25	-2.160×10^{-3}	1.152×10^{-2}	5.029×10^{-3}	-1.388×10^{-4}
0.30	-2.751×10^{-3}	1.208×10^{-2}	7.317×10^{-3}	-5.650×10^{-4}

Table 2 Effect of velocity magnitude on whether the position $Y_{1s} = -0.001$, $Y_{4s} = 0.0003$ is inside or outside the barrier with velocity direction fixed at $Y_{3s}/Y_{2s} = 0.07$, $R_0 = 4.78 \times 10^{-5}$, and $T_p - T_e = 0.04$

Y_{2_s}	0.01	0.009	0.008	0.007	0.006
Y_{3_s}	0.0007	0.00063	0.00056	0.00049	0.00036
τ	0.1245	0.1653	0.1655[a]	0.1665[a]	0.1550[a]
ϕ	34.37°	55.86°	58.29°[a]	54.68°[a]	53.31°[a]
$Y_2(0)$	0.00689	0.00309	0.00280[a]	0.00333[a]	0.00355[a]
R	1.228×10^{-4}	3.55×10^{-5}	0[a]	0[a]	0[a]
Δv^2	0	0	4.50×10^{-7}	1.99×10^{-6}	4.57×10^{-6}
$R < R_0$?	No	Yes	Yes	Yes	Yes

[a]Quantity associated with nearest limiting barrier trajectory (in Δv^2 sense) with $o < R < < R_0$.

are spurious solutions. These results show that the linearized analysis gives excellent accuracy for the values of τ for which the solution has validity. Furthermore, the lower the specific thrust magnitude for the two players, the better are the linearized results.

A number of other barrier trajectories were generated from other points on the BUP, with the results being very similar to those presented here. Thus we can conclude that, in general, the linearized analysis gives excellent results for values of $\tau < \tau_c$.

V. A Method for Locating the Barrier

For the preceding results to have practical utility, they must give information on whether a given state is inside or outside the barrier, and a measure of how close to the barrier this state is located. To use Eqs. (35) to solve this problem, it is convenient to use the interpretation given in Sec. II on barrier trajectories also being minimax range trajectories. If the minimax range trajectory can be found from the desired state, a value of the minimax range R will result. If this is less than the terminal surface radius R_0, then we can conclude that the state is inside the barrier. If R is greater than R_0, the state is outside the barrier.

Conceptually, Eqs. (35) could be solved for $Y_2(0)$, R_0, ϕ, and τ in terms of the desired state Y_1, Y_2, Y_3, and Y_4. However, because of the high decrease of nonlinearity of these equations in ϕ and τ, it is not possible to do this algebraically. Thus a simple search routine was developed to solve this problem. The steps in this routine are as follows:

1) Given Y_{1_s}, Y_{2_s}, Y_{3_s}, and Y_{4_s}, start with $\phi = 0.25°$ and approximate τ by

$$\tau \cong [(Y_{1_s}^2 + Y_{4_s}^2)/(Y_{2_s}^2 + Y_{3_s}^2)]^{1/2} \tag{43}$$

2) Solve the Y_1 and Y_4 equations for R and $Y_2(0)$, with $Y_1 = Y_{1_s}$ and $Y_4 = Y_{4_s}$.

3) Calculate Y_2 and Y_3 from the Y_2 and Y_3 equations.

4) Search on τ to minimize

$$\Delta v^2 = (Y_{2s} - Y_2)^2 + (Y_{3s} - Y_3)^2 \tag{44}$$

5) Go to step 1, and increase ϕ by 0.5° or 1°. Then repeat steps 2-4. Do this through a range of ϕ from about 0° to 360°, discarding results that give $R < 0$.

6) From the preceding data, find the values of ϕ which result in $\Delta v^2 = 0$ with $R \geq 0$ and those for which $R = 0$ and $\Delta v^2 > 0$. If there are two values of ϕ for which $\Delta v^2 = 0$, discard the one with the larger value of R, since this solution generally has $\tau > \tau_c$. Now, for the remaining solution for which $\Delta v^2 = 0$, check τ_c using Eq. (40). If $\tau < \tau_c$ and $R < R_0$, the state is inside the barrier, but, if $\tau < \tau_c$ and $R > R_0$, it is outside the barrier. If $\tau > \tau_c$, the state is inside the barrier with the pursuer capable of intercepting the evader with $R = 0$. No unique optimal strategies exist in this situation. Here a measure of the "velocity distance" at this state is to the nearest barrier trajec-

tory, with $0<R=\epsilon<<R_0$, is the smallest value of Δv^2 attained with $R=0$.

This routine was applied to a number of different initial states. To illustrate the type of results obtained, Table 2 presents the effects of varying the initial velocity magnitude with position fixed at $Y_{1s}=-0.001$, $Y_{4s}=0.0003$, fixed initial velocity direction defined by $Y_{3s}/Y_{2s}=0.07$, and $T_p-T_e=0.04$. For the larger velocities, the state is outside the barrier. As the velocity decreases, the resulting R decreases until the state is inside the barrier. Further decreases in velocity result in the state being inside the barrier with $R=0$. The lower the velocity in this situation, the further is the "velocity distance" Δv^2 from the limiting barrier trajectory with $0<R=\epsilon<<R_0$.

VI. A Proposed Interception Guidance Law

The closed-form solution described previously can be used as the basis for a closed-loop guidance scheme for either an intercepting or an evading spacecraft. Consider the interception problem. Generally, this requires that the intercepting vehicle get within a given distance R_0 of the evading spacecraft. By playing a minimax range, free final time strategy, every nonoptimal maneuver of the evader will result in the reduction of the final range payoff. Optimal play by the pursuer then may force the state from outside to inside the barrier, resulting in successful interception. Such a guidance scheme could be implemented as follows:

1) Use the search method to Sec. V to determine the values of final range R,ϕ, and ϕ associated with the initial state.

2) The thrust direction is determined from Eqs. (33) and (34) using the value of ϕ determined in step 1 and the time-to-go variable τ.

3) As τ decreases, compare the calculated state given by Eqs. (35) with the actual state. When these quantities differ by some small amount, again apply the search method. Note that this can be done very rapidly, since we need only search in small region about the ϕ determined in Step 1. The resulting R should decrease, since any difference between the calculated and actual states should be due mainly to nonoptimal play by the evading vehicle.

4) This procedure, given by the preceding steps, is continued until the minimum range actually is attained by the intercepting vehicle. If the final R is less than R_0, then interception has been attained.

This guidance scheme can also be used by an evading vehicle to avoid interception. In this case, the evader attempts to keep or drive the state outside of the barrier.

VII. Conclusions

Barrier trajectories generated using the closed-form approximate solution agree very closely with trajectories generated using the complete nonlinear dyanmics for values of τ for which the solutions are valid. The accuracy of the closed-form solution decreases with increasing thrusting capability for the two vehicles.

The search technique given in Sec. V uses the closed-form solution to determine whether a given state is inside or outside the barrier and the resulting miss distance that would occur if both vehicles use their optimal minimax range saddle point strategies for the remainder of the game. The closed-loop guidance scheme presented in Sec. VI can be used by either an intercepting or an evading spacecraft. Since only algebraic computations are required, it should be implementable in real time. Furthermore, the feedback, and hence self-correcting, nature of the guidance scheme should allow its use even in state regions where the linearized results are not very accurate.

By suppressing the thrusting capability of the evader, these results are applicable immediately to the problem of intercepting a nonmaneuvering satellite. Similarly, by suppressing the thrusting capability of the pursuer, the results are applicable to the satellite collision avoidance problem.

Appendix: Complete Closed-Form Solution

The complete closed-form solutions for the Y_i in terms of ϕ, τ, R, and $Y_2(0)$ are given by the following equations

$$Y_1=P\cos\tau-Q\sin\tau-K_1\tau-K_2\tau^2/2 -K_3\tau^2/3-K_4\tau^4/4+M \tag{A1}$$

$$Y_2=P\sin\tau+Q\cos\tau+K_1+K_2\tau+K_3\tau^2+K_4\tau^3 \tag{A2a}$$

$$Y_3=-P\cos\tau+Q\sin\tau+K_1\tau+K_2\tau^2/2+K_3\tau^3/3 \tag{A2b}$$

$$+K_4\tau^4/4+N+\beta[(A_2\tau+B_2\tau^2/2 +C_2\tau^3/3+D_2\tau^4/4)] \tag{A3}$$

$$Y_4=2P\sin\tau+2Q\cos\tau-(2K_1+\beta A_2)\tau^2/2 -(2K_2+\beta B_2)\tau^3/6-(2K_3+\beta C_2)\tau^4/12 -(2K_4+\beta D_2)\tau^5/20+(M-N)\tau+S \tag{A4}$$

The parameters in these equations are defined by

$$\beta=T_e-T_p \tag{A5}$$

$$P=(-3R-\beta)\sin\phi+2Y_2(0)\tan\phi-K \tag{A6}$$

$$Q=Y_2(0)-K_1 \tag{A7}$$

$$K_1=\beta(\cos\phi/2-2\cos\phi\sin^2\phi) \tag{A8}$$

$$K_2=\beta(\sin^3\phi+7\sin\phi+6\sin\phi\cos^2\phi) \tag{A9}$$

$$K_3=-\beta(\cos\phi\sin^2\phi-3\cos\phi/4) \tag{A10}$$

$$K_4=-\beta\sin\phi(5/6+\cos^2\phi) \tag{A11}$$

$$M=(4R+\beta)\sin\phi-2Y_2(0)\tan\phi+K_2 \tag{A12}$$

$$N=(-2R-\beta)\sin\phi+Y_2(0)\tan\phi-K_2 \tag{A13}$$

$$S=R\cos\phi-2Y_2(0)+2K \tag{A14}$$

$$A_2=\cos\phi \tag{A15}$$

$$B_2=\sin\phi \tag{A16}$$

$$C_2=-\cos\phi(1+\sin^2\phi)/2 \tag{A17}$$

$$D_2=-\sin\phi(5/6+\cos^2\phi) \tag{A18}$$

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